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Exponential model for option prices: Application to the Brazilian market

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HIGHLIGHTS

- Daily returns on the Ibovespa index follow an exponential distribution.
- Comparison is made between the Black-Scholes and the exponential models for option pricing.
- Near maturity, option prices are better described by the exponential model.
- Possible implications for investment strategies and risk management are briefly discussed.

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ABSTRACT

In this paper we report an empirical analysis of the Ibovespa index of the São Paulo Stock Exchange and its respective option contracts. We compare the empirical data on the Ibovespa options with two option pricing models, namely the standard Black–Scholes model and an empirical model that assumes that the returns are exponentially distributed. It is found that at times near the option expiration date the exponential model performs better than the Black–Scholes model, in the sense that it fits the empirical data better than does the latter model.

1. Introduction

Options are financial instruments that allow their holders the right of buying or selling the underlying asset for a given fixed price, also known as the strike price, on a pre-defined date called the expiration or maturity date. The central issue of modeling option prices is to determine the fair price, also called premium, to pay for a given option before the maturity date, taking into account the statistical distribution of the underlying asset.

The formulation of the Black, Scholes and Merton model in the 1970's established a landmark for option pricing [1]. This model assumes that asset prices in financial markets can be described by a geometric Brownian motion. This hypothesis is the backbone of the so-called *Efficient Market Hypothesis* (EMH), which asserts that the returns of a given stock follow an uncorrelated Gaussian process (white noise).

However, studies on high frequency financial data (e.g., on time scales of the order of a few minutes or less) have shown that price fluctuations behave as non-Gaussian processes [2–4], with the empirical probability distribution function (EDF) of

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returns exhibiting power law tails. On the other hand, on mesoscopic time scales (typically from hours to days) the central part of the EDFs are often better described by an exponential distribution [5–10]. At longer time lags, the EDFs tend to a Gaussian distribution as required by the central limit theorem. Given this scenario, it is natural to ask how option prices should be modeled when the distribution of returns of the underlying asset displays such a variability.

Several non-Gaussian option pricing models have been considered in the literature [11,12]. These include models based on Lévy processes [13,14] and on the so-called q-Gaussian distribution [15], in which cases the distribution of returns have power-law tails; stochastic volatility models [16,17], including the Hull–White and Heston models [18,19]; option pricing with the Edgeworth expansion [20,21]; etc. Of particular relevance to us here is the empirical model for option pricing introduced by McCauley and Gunaratne [22] which assumes an exponential distribution of returns. As mentioned above, financial markets often exhibit exponential distributions on time scales that are comparable to the lifetime of an option, and so the exponential model for option pricing is a natural candidate to describe options in such markets. In contradistinction, power-law distributions tend to be observed in high-frequency data (e.g., intraday quotes) and are thus expected to be less relevant for option pricing in such cases.

In the present paper we perform an empirical analysis of the prices of options on the Ibovespa index of the São Paulo Stock Exchange. First we show that, as in other financial markets [6–9], the central part of the distribution of the Ibovespa returns follows an exponential distribution at mesoscopic times scales. We then proceed to analyze the Ibovespa option market in light of two relevant option pricing models: (i) the standard Black–Scholes model [1] and (ii) the exponential model for option pricing [22] mentioned above. Both models yield an analytical solution for the price of an European call option, which can be easily compared with the quoted market prices. We find that the exponential model gives a better fit to the empirical data for times closer to the option expiration date, whereas for longer periods before expiration the Black–Scholes model offers a better description of the market prices. Our findings thus indicate that the market seems to implicitly take into account the fact that the returns of the Ibovespa (at mesoscopic time scales) follow an exponential distribution. Implications of this finding for possible investment strategies are briefly discussed.

2. Exponential model for option pricing

In this section we collect the main results concerning the exponential distribution for financial returns and its applications to option pricing.

2.1. Distribution of returns

Let us define the logarithmic returns at time lag τ by $x(t) = \ln[S(t+\tau)/S(t)]$, where $S(t)$ is the price of the relevant financial asset at time t and τ is the time lag. The exponential distribution, $f(x, t)$, of the log-returns x is defined by [22]

$$f(x,t) = \begin{cases} Ae^{\gamma(x-\delta)} & \text{if } x \le \delta \\ Be^{-\nu(x-\delta)} & \text{if } x > \delta, \end{cases} \tag{1}$$

where δ, γ and ν are parameters that characterize the distribution. From the normalization condition, $A/\gamma + B/\nu = 1$, and by imposing $\langle x \rangle = \delta$, one finds that $A/\gamma^2 = B/\nu^2$. One can also show that the variance of the exponential distribution is given by $2(\gamma\nu)^{-1}$; see Ref. [22] for details.

Let us now recall that the folded cumulative distribution, $G(x)$, associated with a probability density function $f(x)$ is defined by

$$G(x) = \begin{cases} F(x), & \text{if } F(x) \le \dfrac{1}{2} \\ 1 - F(x), & \text{otherwise} \end{cases} \tag{2}$$

where $F(x)$ is the cumulative distribution: $F(x) = \int_{-\infty}^{x} f(x)\mathrm{d}x$. Note that $G(x)$ for the exponential distribution given in (1) is also a bilateral exponential function. This means, in particular, that in a semi-log plot the function $G(x)$ for the exponential distribution has a tent-like shape, in contradistinction to a Gaussian distribution whose folded cumulative distribution has a downward concavity.

In Fig. 1(a) we plot in a semi-log scale the empirical folded cumulative distribution of the Ibovespa returns for $\tau = 1, 5$ and 20 days. These distributions were generated from the historical time series of the daily closing prices of the Ibovespa, from its inception in January 1968 up to February 2004, totaling 8889 data points. One sees from this figure that the empirical distributions (solid lines) deviate from a Gaussian (dashed line) and follow instead an exponential law, particularly in the central region of the EDF which displays the tent-like shape typical of an exponential distribution.

At shorter time scales, the distribution of returns of the Ibovespa becomes heavy-tailed as seen in Fig. 1(b), where we plot the folded cumulative distribution of intraday returns for $\tau = 15$, 60, and 180 min. The distributions in this figure were constructed from a time series of 19995 intraday quotes at every 15 min covering the years from 1998 to 2001. Note, in particular, that for $\tau = 15$ min the distribution has an upward concavity typical of power law distributions. However, as τ increases the empirical distribution evolves towards an exponential distribution, as seen in the case for $\tau = 180$ min

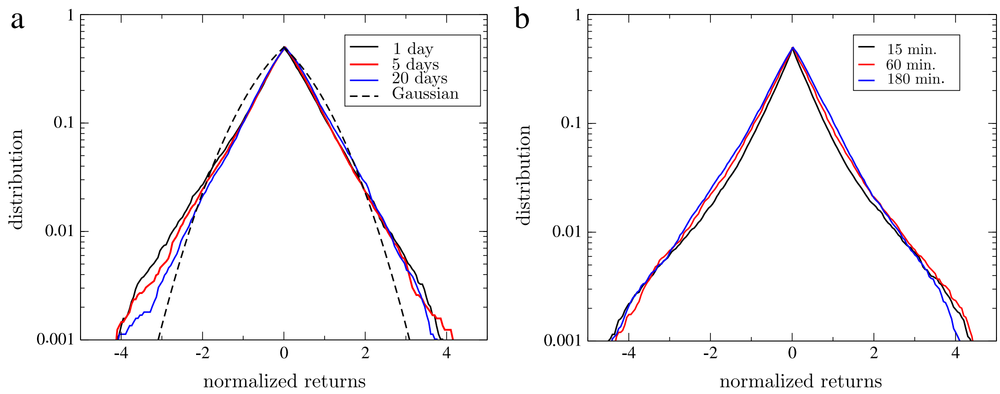

Fig. 1. Folded cumulative distribution of the Ibovespa returns. The returns were normalized by the standard deviation. (a) shows the distributions of returns with time lags of 1, 5 and 20 days, whereas (b) shows the distributions of intraday returns at time lags of 15, 60 and 180 min.

where the distribution already shows a tent-like shape. Analysis of Fig. 1 then reveals that the Ibovespa returns follow an exponential distribution for time lags varying from 3 h up to 20 days. This observation was the main motivation for our investigation of the Brazilian option market vis-a-vis the exponential model for option pricing.

2.2. Option pricing

We recall that one of the assumptions of the Black–Scholes model is that the price $S(t)$ of the underlying asset price follows a geometric Brownian motion, which implies that the distribution of the log-returns is a Gaussian. (Because of this, we shall often refer to the Black–Scholes model as the Gaussian model for option pricing.) Based on this assumption, the Black–Scholes formula [1] for option prices is given by

$$C_{BS}(S,K,r,t;\sigma) = SN(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2), \tag{3}$$

where the function $N(x)$ is the cumulative distribution of a normal random variable

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y, \tag{4}$$

and

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{5}$$

$$d_2 = \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{6}$$

Here T is the expiration date, K is the strike price, r is the daily interest rate, and σ is an unknown parameter called volatility.

In the empirical option pricing model introduced by McCauley and Gunaratne [22], it is assumed that the log-returns of the underlying asset follow an exponential distribution. For this reason we shall refer to this model as the *exponential model for option pricing*. The exponential model also yields an explicit formula for the fair price $C(S,K,r,t)$ of a European option [22], given by the following expression:

$$C\,\mathrm{e}^{r\Delta t} = \begin{cases} S\mathrm{e}^{\delta}\dfrac{\gamma^2(\nu-1)+\nu^2(\gamma+1)}{(\gamma+\nu)(\gamma+1)(\nu-1)} + \dfrac{K\gamma}{(\gamma+1)(\gamma+\nu)}\left(\dfrac{K}{S}\mathrm{e}^{-\delta}\right)^{\gamma} - K, & S > K\mathrm{e}^{-\delta} \\ \dfrac{K\nu}{(\nu-1)(\gamma+\nu)}\left(\dfrac{K}{S}\mathrm{e}^{-\delta}\right)^{-\nu}, & S < K\mathrm{e}^{-\delta}. \end{cases} \tag{7}$$

The parameters δ, γ and ν are related by the risk-neutral condition:

$$r = \frac{1}{\Delta t}\int_t^T \mu(s)\mathrm{d}s = \frac{1}{\Delta t}\left(\delta + \ln\left(\frac{\gamma\nu + (\nu-\gamma)}{(\gamma+1)(\nu-1)}\right)\right), \tag{8}$$

where μ is the theoretical expected return, which can be computed in terms of the parameters of the exponential distribution (1) leading to (8); see Ref. [22]. Since the interest rate r is assumed to be known one can solve this expression for δ, so that we end up with only two unknown parameters, namely, γ and ν.

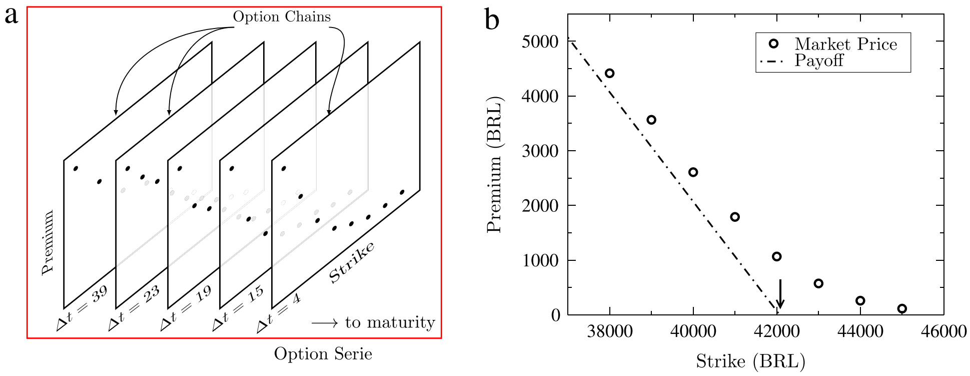

Fig. 2. (a) Schematics explaining the concept of *option chains*; (b) the option chain for the IBOVL option series at 15 days before expiration. The dash–dot line represents the intrinsic value of the option, corresponding to the difference between the current price (arrow) of the Ibovespa and the strike price.

3. Methodology

3.1. Option data

We study options whose underlying asset is the Ibovespa index, which is the main stock index of the São Paulo Stock Exchange. Analysis of the options on the Ibovespa thus offers a general idea about the behavior of the Brazilian option market. In the present study we analyzed the closing market prices of (European) call options on the Ibovespa traded daily in the years 2005 and 2006. Before we describe the data, let us first introduce some terminology about options on the Ibovespa.

A set of options that have the same date of expiration is called an *option series*. Option series on the Ibovespa are denoted by the symbol IBOV followed by a letter indicating the date of expiration according to the following convention: the letters from A to L indicate call options expiring on the months from January to December, respectively. Options on the Ibovespa index always expire on the Wednesday closest to the 15th day of the month, and so it suffices to specify the month to fix the expiration date. For instance, all options belonging to the series denoted by IBOVD expire on the Wednesday closest to April 15th, following the launch of the series. Once an option series is authorized by the Exchange, trading on the options of that series begins with investors taking short and long positions.

We shall denote by an *option chain* the set of options belonging to the same option series that are traded on a particular day. Thus, for each option chain we have a set of premiums (closing market prices) as a function of the strike price; see Fig. 2(a) for a schematic illustrating option chains. An example of a specific option chain for the option series IBOVL for the year 2006 at a time corresponding to 15 days before expiration is shown in Fig. 2(b).

Options on the Ibovespa expire on even months, hence there were 12 option series in the two-year period (2005–2006) analyzed here, with a total of 850 option chains. When a new option series is launched by the Exchange, typically 60 days before the expiration date, trading is authorized on several options with different strikes near the then-current market price of the Ibovespa. The number of strikes that are actually traded at each particular day varies considerably during the lifetime of an option series. This is illustrated in Fig. 3 where we show the average number (among our 850 options chains) of strikes traded as a function of the time to maturity. One sees that shortly after an option series is launched (i.e., for large Δt) trading is limited to a few strikes. As time passes (and Δt decreases), the number of strikes traded tends to grow and reaches an average number around five or more. Then, close to expiration (i.e., for small Δt) the number of strikes traded decreases considerably, as trading concentrates on those strikes that are closer to the current value of the Ibovespa.

In our empirical analysis described below, we used only option chains that had at least four strikes negotiated. Out of our total set of 850 option chains, 441 of them satisfied this criterion. Options chains with less than four traded strikes were discarded because the small number of data points renders the analysis less reliable. For each of these selected option chains we performed a least-square fit of the empirical data (market price *vs.* strike) by the two option-pricing formulas predicted by the Gaussian and the exponential models, respectively. Using a suitable figure of merit for the fits (see below), we then compared which model gives a better fit to the data as a function of time to maturity.

3.2. Empirical analysis

Our main object of study here is the set of option chains on the Ibovespa index. As explained above, each option chain consists of a list of premiums C_i with its respective strike prices K_i, all traded in a particular day t before the expiration date T.

The Black–Scholes model states that all options belonging to an option chain should have the same volatility σ, but options with different strike prices are negotiated separately and so their volatility may vary according to the investor valuations. Investors determine the volatility in two ways: (i) the historical method, which estimates statistically the volatility over the history of the underlying asset prices, and (ii) the "implied volatility" method that consists of finding the volatility σ_{imp} of the Black–Scholes formula that yields the quoted market price of an option, i.e., $C_{\text{BS}}(S, K, r, \Delta t; \sigma_{\text{imp}}) =$

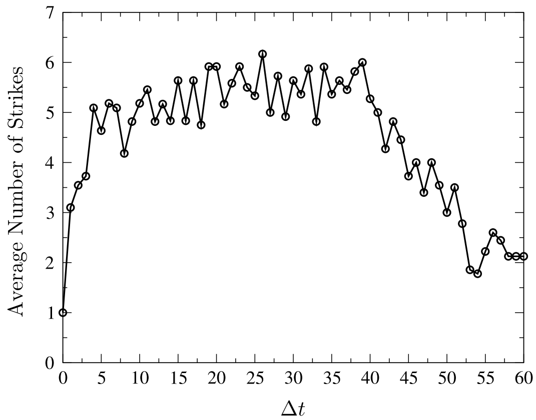

Fig. 3. Average number of strikes in an option chain as a function of the time to maturity.

C_{market}. In the first method (which relies only on the underlying asset prices) the volatility is the same for all options on that asset, whereas in the second method each option yields a value for the implied volatility which generally differs for different values of the strike price, giving rise to the so-called volatility smile effect.

Here we adopt a somewhat intermediate approach in that we shall assume that the volatility σ is the same for all options of a given option chain. We obtain σ by performing a least-square fitting of the formula $C_{\text{BS}}(S, K, r, \Delta t; \sigma)$ given in (3) to the empirical data corresponding to the option premium versus strike price. More specifically, for any given option chain we find the parameter σ that minimizes the residual sum of squares:

$$\frac{\partial}{\partial \sigma}[SS_{res}] = 0, \tag{9}$$

where

$$SS_{res} = \sum_{i=1}^{N} (C_i - C_{\text{BS}}(S, K_i, r, \Delta t; \sigma))^2 . \tag{10}$$

Here N is the total number of strikes in the option chain, C_i is the "empirical" market price of the option, and K_i is the respective strike price. The other parameters are the time to maturity $\Delta t = T - t$, where T is the expiration date and t is the day when the option was negotiated, the asset price $S(t)$ at time t, and the daily interest rate r, which is assumed to be the reference rate DI (Interbank Deposit rate) extracted from Ref. [23]. The numerical method used here to compute the value σ that minimizes SS_{res} is the so-called golden section method [24].

A similar analysis can be performed with the exponential model for option pricing. In this case, one seeks to adjust the theoretical formula (7) predicted by the exponential model to the empirical data of the chain. This can easily be done by computing the unknown parameters μ and ν that minimize the corresponding residual sum of squares SS_{res}, where SS_{res} is as defined in (10) but with the difference that (7) now replaces the Black–Scholes formula. For this minimization procedure we have used the Downhill Simplex method [24] for two dimensions.

The fitting procedure described above has the advantage that it allows a direct comparison between the two models since we can easily compare which theoretical formula yields a better fit to the empirical data. For example, in Fig. 4 we show the respective fits given by the Gaussian model (solid curve) and the exponential model (dashed curve) for the option chain of the series IBOVL of year 2006 at 15 days before expiration. To estimate the error in the volatility σ for the Gaussian model and in the parameters γ and ν for the exponential model, we used the bootstrap method where we resampled the data (with repetition) 2000 times. For each bootstrap sample we performed the corresponding fitting procedures described above and then computed the average and standard deviation of the fitting parameters. For the data shown in Fig. 4 the Gaussian model yields $\sigma = 0.0135 \pm 0.0005$, whereas for the exponential model we found $\gamma = 20.2 \pm 1.4$ and $\nu = 30.7 \pm 1.6$. A visual inspection of Fig. 4 reveals that the theoretical curve predicted by the exponential model seems to adjust better this particular set of data.

This comparison can be made more quantitative by introducing the coefficient of determination R^2 defined as

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}}, \tag{11}$$

where SS_{tot} is the total sum of squared differences from the mean (proportional to the variance of the data):

$$SS_{tot} = \sum_{i=1}^{N} \left(C_i - \bar{C}\right)^2 , \tag{12}$$

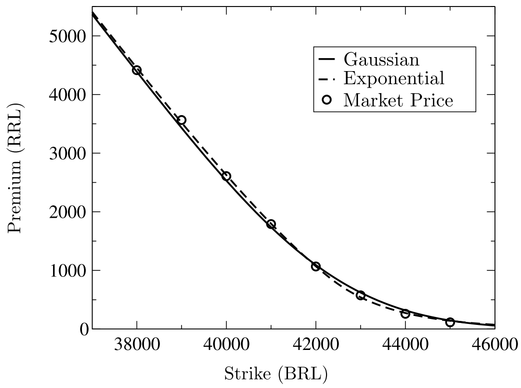

Fig. 4. Fits of the empirical option data shown in Fig. 2(b) by the Gaussian model (solid curve) and the exponential model (dashed curve). The parameters of the exponential distribution are $\gamma = 20.2 \pm 1.4$ and $\nu = 30.7 \pm 1.6$, whereas for the Gaussian model the volatility is $\sigma = 0.0135 \pm 0.0005$. The errors in the parameters were estimated using the bootstrap method; see text.

with $\bar{C}$ denoting the mean of the option premiums of the corresponding option chain. The coefficient of determination R^2 ranges from 0 to 1 and indicates how well the data is fitted by the model curve: the closer R^2 is to unity the better is the agreement between the theoretical curve and the empirical data. Strictly speaking, the coefficient of determination R^2 is not mathematically guaranteed to be in the range from 0 to 1 for nonlinear regression; nonetheless, it can still be used as a relevant measure of the goodness of the fit.

For example, in Fig. 4 we have $R_g^2 = 0.9982(3)$ for the Gaussian model and $R_e^2 = 0.99974(4)$ for the exponential model. Here the errors in R^2 were estimated via the bootstrap method described above. As $R_g^2 < R_e^2$, we conclude that the exponential model does indeed give a better description of the data, in agreement with our visual inspection of Fig. 4. It is interesting to note that both models describe relatively well the empirical data shown in Fig. 4 without the need of a volatility smile, i.e., using the same model parameters for the entire option chain. This agreement between the two models is reflected in the fact that both of them yield values of R^2 quite close to 1 (see above), although the exponential model performs slightly better in this case. For other option chains – particularly close to the expiration date – the two models differ more considerably. To investigate how the Black and Scholes and the exponential models behave as a function of the time to maturity we applied the fitting procedure described above to our entire set of selected option chains, as discussed in the next section.

4. Results and discussions

The 441 options chains selected for our analysis are spread over a wide range of times before expiration, varying from 59 days before expiration until the day before maturity. This allowed us to study which of the two models above gives a better fit to the empirical option data as a function of the time to maturity. To do this, we applied the fitting procedures described in Section 3.2 to each of the selected option chains and compared the resulting coefficients of determination, R_g^2 and R_e^2, for the Gaussian and exponential models, respectively.

Fig. 5(a) shows the difference $R_\text{g}^2 - R_\text{e}^2$ versus the time to maturity Δt. The quantity $R_\text{g}^2 - R_\text{e}^2$ leads to the following interpretation: a *negative* outcome means that the exponential model fits the data *better* than does the Gaussian model; conversely, a positive difference means that the Gaussian model performs better than the exponential model in the sense that it yields a better fit to the data. Thus, all points below the red line (horizontal axis) in Fig. 5(a) correspond to option chains for which the exponential model provided a better fit to the empirical data, whereas points above this line indicate the opposite, i.e., the Gaussian model gave a better fit.

One interesting fact in Fig. 5(a) is that the exponential seems to adjust better the empirical data near the expiration date, whereas the Gaussian model works better for longer times before maturity. To see this, notice that close to expiration, say, less than ten days to maturity, there are more points below zero; whereas the opposite is true for intermediate values of Δt, say, from 20 to 45 days before expiration, in which case the majority of points lie above zero.

To make this analysis more precise, we computed for each Δt the percentage of times that the exponential model gives a better fit to the data in comparison to the Gaussian model. This is indicated in Fig. 5(b) by the red squares. One sees from this figure that the exponential model gives a better fit to the empirical data in the majority of cases (i.e., above the 50% line) for each Δt up to 7 days prior to maturity. For longer times prior to maturity, Fig. 5(b) shows that the Gaussian model adjusts better the data in the majority of cases for most days up to 45 days before the expiration date. For $\Delta t > 45$ days there is no clear indication as to which model is best as the points (squares) in Fig. 5(b) fluctuate widely. This is in part due to the fact that at such long times before expiration, i.e., shortly after the option series is launched, there are few strikes traded at each particular day which makes the fitting procedure somewhat less reliable for these options chains.

To mitigate the fluctuations seen in the daily percentage measure shown in Fig. 5(b), we computed the cumulative frequency that gives the percentage of times that the exponential model performs better than the Gaussian model when

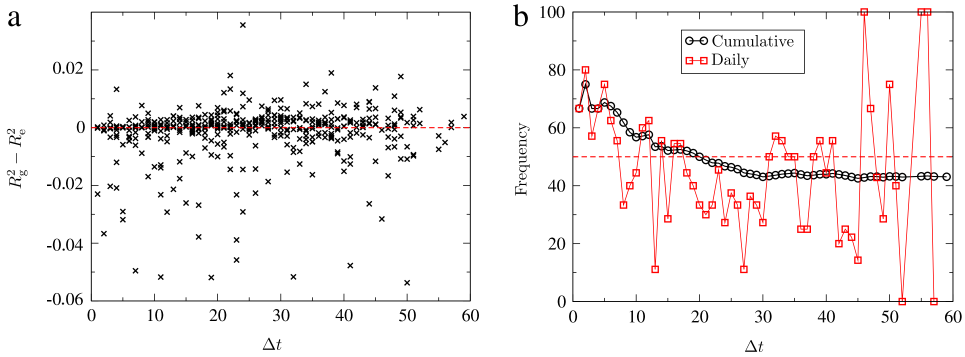

Fig. 5. (a) Difference between $R_g^2 - R_e^2$ as a function of the time Δt to maturity. Points below zero (horizontal dashed line) correspond to option chains for which the exponential model fits the data better than does the Gaussian model. Conversely, points above zero represent option chains where the Gaussian model performed better than the exponential model. (b) Percentage of option chains better fitted by the exponential model as a function of the time Δt to maturity. The red squares are the results for each Δt and the black circles are the cumulative frequency from 0 to Δt; see text. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

considering all option chains up to a time Δt. In other words, for a given Δt, we consider all options chains in the period from 0 to Δt and then compute the (relative) number of times that the exponential model gives a better fit to the data. This cumulative frequency is shown in Fig. 5(b) by open (black) circles. We see from this plot that for any period of time up to 20 days before the expiration date, the exponential model adjusts better the majority of option chains within this period.

Notice, in particular, that at around $\Delta t = 20$ the cumulative frequency crosses the 50% line (red dashed line). This means that for periods longer than 20 days the Gaussian model performs better than the exponential model in the sense that it gives a better fit to the data in the majority of cases within the period considered. In fact, it is noteworthy that 57% of all option chains studied here are best fitted by the Gaussian model. As most investors use the Black–Scholes model as a reference to the fair price of options, it is reasonable to expect that option market prices tend to follow this model in the overall majority of cases. However, near maturity the exponential model performs better than the Black–Scholes model, as discussed above.

The results shown in Fig. 5(b) reveal that up to 20 days prior to maturity the market option prices are better fitted by the exponential model. This agrees with the result of Section 2.1 where it was shown that the empirical distribution of the Ibovespa returns follows an exponential distribution for time lags up to 20 days. Our findings thus seem to indicate that the Brazilian option market implicitly takes into account the fact that the returns of Ibovespa follow an exponential distribution.

These findings suggest an influence of the non-Gaussian behavior of the underlying asset on the option price. So in a situation where traders are quite actively seeking to make profits from traded options at times near the expiration of the contract, it is quite plausible that the prices of these options do not follow a Gaussian-based model. This can be of significance to construct possible investment strategies.

5. Conclusions

We have analyzed the Brazilian option market in light of the Gaussian and exponential models for option pricing. By fitting the empirical data with the respective pricing formulas predicted by these models, we were able to compare which model performs better as a function of the time to maturity. We found that near maturity the majority of the option chains traded were better fitted by the exponential model. This result is in agreement with the fact that the distribution of daily returns of the Ibovespa follows an exponential distribution.

In other words, the exponential distribution for the Ibovespa returns does seem to have a direct impact on the Brazilian option market which is well captured by the exponential model for option pricing. In contrast, option models based on distributions with tails that go to zero slower than exponential, such as power-law distributions, are expected to be less relevant given that these heavy-tailed distributions occur in time scales (e.g., intraday quotes) much shorter than the typical lifetime of an option. It should also be noted that the exponential model describes relatively well the market prices in a given option chain without the need of volatility smiles.

The results obtained from our fitting procedure demand a more practical test, such as backtesting of historical data, to further compare the outcome of both models. In this context, the possibility of developing investment strategies based on the exponential model deserves to be investigated in detail.

The results reported here should also have implications for the risk management of derivative contracts, given that the exponential distribution provides a more conservative estimation of extreme events than the Gaussian distribution. Furthermore, the analytical formula for the option price provided by the exponential model should allow the computation of risk measures such as VaR and expected shortfall.

It is worth emphasizing that the Ibovespa index reflects the average behavior of stock prices in the Brazilian financial market. The results reported here – although based on options on the Ibovespa – should therefore reflect the generic behavior of the Brazilian option market. It would then be of interest to perform a similar analysis of options on individual stocks to verify whether the exponential model applies to these options as well. Furthermore, given that the exponential distribution has been observed in several other financial markets, it is expected that the exponential model for option pricing should also apply to these markets. It is thus hoped that the present work will stimulate further research on these directions.

Acknowledgments

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Investigation of non-Gaussian effects in the Brazilian option market

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HIGHLIGHTS

- Non-Gaussian effects are observed in the Brazilian stock and options markets.
- The exponential, q-Gaussian, and Black–Scholes option pricing models are compared.
- Near maturity, option prices are much better described by the exponential model.
- The exponential model can better describe the skewed volatility smiles.
- Possible implications of our results for investment strategies are briefly discussed.

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ABSTRACT

An empirical study of the Brazilian option market is presented in light of three option pricing models, namely the Black–Scholes model, the exponential model, and a model based on a power law distribution, the so-called q-Gaussian distribution or Tsallis distribution. It is found that the q-Gaussian model performs better than the Black–Scholes model in about one third of the option chains analyzed. But among these cases, the exponential model performs better than the q-Gaussian model in 75% of the time. The superiority of the exponential model over the q-Gaussian model is particularly impressive for options close to the expiration date, where its success rate rises above ninety percent.

1. Introduction

Options are important financial instruments that can be used for both investment strategies and risk management and they represent nowadays a multi-trillion market. An option is a derivatives contract that gives its holder the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a specified strike price on a specified maturity date. Because the price of an option depends not only on the spot price of the underlying asset but also on the intensity of its fluctuations (i.e., the volatility), it is paramount to have a good model for the underlying asset price dynamics in order to obtain a reliable model for the option fair price.

The standard model of finance – the Black–Scholes model – surmises that risky asset prices can be described by a geometric Brownian motion, implying that the asset's logarithmic returns follow an uncorrelated Gaussian process. Within this Gaussian framework, an analytical expression for the price of a European call option – the celebrated Black–Scholes formula – can be obtained [1]. In the last two decades or so, however, empirical evidence has accumulated showing that

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financial markets often display heavy-tailed distributions [2–4], meaning that large price fluctuations occur more frequently that predicted by Gaussian statistics. These findings led to the necessity to consider non-Gaussian distributions as alternative models to describe price fluctuations [5,6].

One such distribution that has attracted considerable attention is the so-called Tsallis distribution [7,8], also known as the q-Gaussian distribution, which has the interesting feature that it decays with power law tails. (The q-Gaussian recovers the standard Gaussian distribution for $q = 1$.) An option pricing model within the framework of the q-Gaussian distribution has been introduced by Borland [9,10] who derived an analytic expression for the option price which generalizes the Black–Scholes formula. Another important non-Gaussian option pricing model is the empirical model introduced by McCauley and Gunaratne [11] which assumes that the returns follow an exponential distribution. Other non-Gaussian approaches to pricing options include models based on Lèvy stable distributions [12,13], which also have power law tails, and the so-called stochastic volatility models where the volatility of the underlying asset price is regarded as a randomly fluctuating quantity [14,15]. Option pricing strategies based on a variational minimization of the risk over the duration of the option have been used in [16,17].

Another important stylized fact of financial data is a phenomenon akin to intermittency in turbulent flows [3,18,19]. Fluid intermittency is characterized by the tendency of the distribution of velocity differences between two points to develop long non-Gaussian tails at short distances. Similarly, in financial markets intermittency is manifest in the fact that the empirical probability density functions (pdf) of price returns – i.e., the logarithmic differences between prices at two instants separated by a given time lag – often depend on the time lag. For time lags of the order of minutes or less the empirical pdfs tend to display power law tails, whereas for lags of the order of hours or a few days the central part of the pdf is better described by an exponential distribution, with a Gaussian regime being recovered for longer time scales; see, e.g., Refs. [20–24] for discussions of this phenomenon. The change in form of the empirical return distributions from an exponential law at the daily scale to a Gaussian distribution for larger time lags has also been studied in the context of the Heston model for stochastic volatility [25]. The convergence from power-law tails to a Gaussian distribution has been empirically investigated in several stock indexes, such as the Dow Jones and the NYSE indexes [26]. Intermittency effects in finance have also been studied by means of the Kramers–Moyal coefficients associated with the evolution equation for the probability density function of the price returns, see, e.g., Refs. [27–29].

Intermittency effects pose a serious problem for option pricing: since options have a lifespan of a couple of months but are frequently traded on relatively short-time scales, it is not clear *a priori* which type of model distribution one should use to price options in markets where the empirical pdfs vary considerably with the time scale. It is thus important to pursue empirical analyses of option markets in light of different pricing models—Gaussian and non-Gaussian ones. In this context, it is of particular interest to investigate how these pricing models fare with respect to the time to maturity. For instance, in a recent comparative study of the exponential and the Black–Scholes models as applied to the Brazilian option market [24], it was found that close to maturity the exponential model performs better than the Black–Scholes model.

In the present paper we investigate the applicability of the q-Gaussian model to the Brazilian market. First we analyze the statistics of the Ibovespa index, which is the main stock index of the São Paulo Stock Exchange. We study historical series of both daily closing prices and intraday quotes at 15 min intervals. We observe that the empirical distribution of the intraday returns is well described by a q-Gaussian distribution, whereas the daily returns follow an exponential distribution. After detecting this intermittency effect in the Ibovespa, we then proceed to analyze the option market on the Ibovespa index by studying a set of 345 option chains covering a period of two years of trading.

First we compared the q-Gaussian model (with $q > 1$) to the Black–Scholes formula ($q = 1$) and found that the former provides an improvement over the latter in only about 30% of the cases. We then applied the exponential model to the option chains for which the q-Gaussian model surpasses the Black–Scholes model. Here we found that the exponential model fits better the data in 75% of the cases, implying that the q-Gaussian model performs simultaneously better than the Black–Scholes and the exponential models in less than 10% of all option chains analyzed here. Furthermore, we observe that the exponential model works significantly better than the q-Gaussian model for option chains close to the expiration date, confirming a trend (in favor of the exponential model near maturity) that was seen in a previous comparison [24] between the exponential model and the Black–Scholes model, as mentioned above.

We argue that part of the superiority of the exponential model is due to the fact that the behavior of the market implied volatility as function of the strike price is highly asymmetrical for option chains near maturity. The q-Gaussian model, being symmetric, cannot describe such asymmetric patterns, whereas the exponential model is inherently asymmetric and hence can better cope with skewed volatility smiles. Although our results are based on the Brazilian option market, we expect that they should apply to other financial markets where exponential distributions for asset returns have been observed.

The paper is organized as follows. In Section 2 we present a brief review of the three option pricing models that will be considered here, namely the Black–Scholes model, the exponential model, and the option model based on the q-Gaussian distribution. In Section 3 an empirical analysis of both the daily and intraday returns of the Ibovespa is performed in light of these three models. In Section 4 we briefly described our option data and the methodology used to study them. The results of our analysis of the Brazilian option market are presented in Section 5. In Section 6 we summarize our main findings and conclusions.

2. Option pricing models

In order to make this paper self-contained, we present here a brief review of the three option pricing models that will be considered in our study, namely the Black–Scholes, the exponential, and the q-Gaussian models. In each of these three cases, an analytical formula for pricing a European call option is available, as will be discussed below. (Readers who are knowledgeable about these models may wish to skip this section and proceed directly to our empirical analysis that starts in Section 3.)

Let C denote the price at time t of a European call option written on an underlying asset of price S, with strike price K and expiration at time T. In the risk-neutral valuation scheme the option price can be written as the discounted price of the expected future payoff:

$$C(S, K, \Delta t) = e^{-r\Delta t}\langle \max(S(T) - K, 0)\rangle \tag{1}$$

where $\Delta t = T - t$ is the time to expiration, r is the risk-free interest rate, and the bracket $\langle\cdot\rangle$ means average over $S(T)$ using the risk neutral measure. In discussing the option models under study here, we shall only quote the respective price formulas for each model; further details about their derivations can be found in the accompanying references.

2.1. The Black–Scholes model

In the Black–Scholes model the price $S(t)$ of a risky asset (say, a stock or a stock index) is assumed to follow a geometric Brownian motion described by the following stochastic differential equation

$$dS(t) = \mu S dt + \sigma S dW(t), \tag{2}$$

where μ is the expected return rate, σ is the volatility, and $W(t)$ is the standard Wiener process. Here both μ and σ are assumed to be constant for simplicity. If we define the logarithmic return $Y(t)$ as the process

$$Y(t) = \ln S(t) - \ln S(0) \tag{3}$$

it then follows that $Y(t)$ is a Brownian motion:

$$dY(t) = \mu dt + \sigma dW(t) \tag{4}$$

with $Y(0) = 0$. Hence the pdf of $Y(t)$ is a Gaussian distribution:

$$p(Y, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{\left[Y - (\mu - \frac{1}{2}\sigma^2)t\right]^2}{2\sigma^2 t}\right). \tag{5}$$

When performing empirical analyses of returns (see Section 3), it is often convenient to normalize the returns to unit variance, and so we define a new variable

$$x = \frac{Y}{\sqrt{\mathrm{Var}[Y]}} = \frac{Y}{\sqrt{\sigma^2 t}}, \tag{6}$$

whose pdf then becomes

$$p(x, t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x - \mu')^2}{2}\right), \tag{7}$$

where

$$\mu' = \frac{\left(\mu - \frac{1}{2}\sigma^2\right)\sqrt{t}}{\sigma}.$$

One then sees that the normalized mean μ' is proportional to $\sqrt{t}$, which becomes negligibly small for small timescales. Because of this property, we shall often assume that the normalized returns at short time lags have zero mean, which is a valid approximation for the empirical data, as we will see later.

To price options in the above Gaussian framework, one can use (1) with the appropriate risk neutral measure which in this case is obtained by setting $\mu = r$ in (2). This leads to the famous Black–Scholes formula [1] for the option price:

$$C(S, K, \Delta t; \sigma) = S\,N(d_1) - K\,e^{-r\Delta t}N(d_2) \tag{8}$$

where $N(x)$ is the cumulative distribution of a normal random variable

$$N(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy \tag{9}$$

and

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)\Delta t}{\sigma\sqrt{\Delta t}} \tag{10}$$

$$d_2 = \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right)\Delta t}{\sigma\sqrt{\Delta t}}. \tag{11}$$

In the left hand side of (8) we have explicitly indicated the dependency of the price C on the volatility σ, which will be treated as a free parameter to be determined from the market prices, as explained in Section 4.

2.2. Exponential model

In the exponential model proposed by McCauley and Gunaratne [11], the returns are assumed to follow an exponential distribution of the form

$$p(x) = \begin{cases} A\, e^{\gamma(x-\delta)} & \text{if } x < \delta \\ B\, e^{-\nu(x-\delta)} & \text{if } x > \delta, \end{cases} \tag{12}$$

where A, B and δ are constants. By imposing the normalization of $p(x)$, together with the condition that $\langle x \rangle = \delta$, one obtains

$$A = \frac{\gamma^2}{\gamma + \nu} \tag{13}$$

and

$$B = \frac{\nu^2}{\gamma + \nu}. \tag{14}$$

The variance of the above exponential distribution is found to be

$$\mathrm{Var}[x] = \frac{2}{\gamma\nu}, \tag{15}$$

so that the distribution (12) can be easily normalized to unit variance by setting

$$\gamma\nu = 2. \tag{16}$$

A closed formula for a European call option can also be obtained under the assumption that the returns of the underlying asset follow the exponential distribution given in (12). As shown by McCauley and Gunaratne [11], the option price $C(S, K, \Delta t; \gamma, \nu)$ in this model is given by

$$C\, e^{r\Delta t} = \begin{cases} Se^{\delta}\dfrac{\gamma^2(\nu-1) + \nu^2(\gamma+1)}{(\gamma+\nu)(\gamma+1)(\nu-1)} + \dfrac{K\gamma}{(\gamma+1)(\gamma+\nu)}\left(\dfrac{K}{S}e^{-\delta}\right)^{\gamma}, & S > Ke^{-\delta} \\ \dfrac{K\nu}{(\nu-1)(\gamma+\nu)}\left(\dfrac{K}{S}e^{-\delta}\right)^{-\nu}, & S < Ke^{-\delta}. \end{cases} \tag{17}$$

Using the risk-neutral condition, whereby the expected stock price $\langle S(t) \rangle$ should behave as the risk free investment, one has

$$r = \frac{1}{\Delta t}\int_t^T \mu(s)ds, \tag{18}$$

where μ is the expected rate of return of the underlying asset. Using the distribution (12) one finds [11]

$$r = \frac{1}{\Delta t}\left[\delta + \ln\left(\frac{\gamma\nu + (\nu - \gamma)}{(\gamma+1)(\nu-1)}\right)\right]. \tag{19}$$

This equation allows one to write δ in terms of γ, ν, and the risk-free interest rate r. Thus, the exponential model for option pricing contains two unknown parameters, namely, γ and ν, which need to be determined from the market prices; see Section 4.

2.3. The q-Gaussian model

In the non-Gaussian option model considered by Borland [9,10], it is assumed that the log returns, $Y(t) = \ln S(t+\tau) - \ln S(\tau)$, follow a process given by

$$dY = \mu dt + \sigma d\Omega(t), \tag{20}$$

where

$$d\Omega(t) = P_q(\Omega, t)^{\frac{1-q}{2}} dW(t), \tag{21}$$

with q being a real number. Solving the Fokker–Planck equation associated to (21) with the initial condition $\Omega(0) = 0$, one finds that the corresponding probability density $P_q(\Omega, t)$ is given by a q-Gaussian or Tsallis distribution:

$$P_q(\Omega, t) = \frac{1}{Z_q(t)}\left[1 - \beta(t)(1-q)\Omega^2\right]^{\frac{1}{1-q}}, \tag{22}$$

where

$$Z_q(t) = \left[(2-q)(3-q)c_q t\right]^{\frac{1}{3-q}} \tag{23}$$

and

$$\beta(t) = \frac{c_q}{Z_q(t)^2}, \tag{24}$$

with

$$c_q = \frac{\pi \Gamma\left(\frac{3-q}{2q-2}\right)^2}{(q-1)\Gamma\left(\frac{1}{q-1}\right)^2}. \tag{25}$$

Note that in the limit $q \to 1$ the distribution $P_q(\Omega, t)$ shown in (22) yields a Gaussian, as it should since in this limit the process $\Omega(t)$ becomes a standard Brownian motion; see (21). For $q > 1$, on the other hand, the q-Gaussian has power law tails

$$P_q(\Omega, t) \sim \frac{1}{\Omega^{\frac{2}{q-1}}}, \qquad |\Omega| \to \infty, \tag{26}$$

as one clearly sees from (22). The variance of the q-Gaussian distribution is given by

$$\langle \Omega(t)^2 \rangle = \frac{1}{(5-3q)\beta(t)} \tag{27}$$

which diverges for $q \geq 5/3$. Hence we shall limit ourselves to the case $1 \leq q < 5/3$.

As defined in (20), the process $Y(t)$ is related to the process $\Omega(t)$ via

$$\Omega(t) = \frac{Y(t) - \mu t}{\sigma}. \tag{28}$$

It then follows from (22) that the probability density of $Y(t)$ is

$$P_q(Y, t) = \frac{1}{Z(t)}\left[1 - \frac{\beta(t)}{\sigma^2}(1-q)(Y - \mu t)^2\right]^{\frac{1}{1-q}}, \tag{29}$$

whose variance is

$$\langle (Y - \mu t)^2 \rangle = \frac{\sigma^2}{(5-3q)\beta(t)}. \tag{30}$$

If one now considers normalized returns, $x = (Y - \mu t)/\sqrt{\langle (Y - \mu t)^2 \rangle}$, one has

$$P_q(x) = \frac{\Gamma\left(\frac{1}{q-1}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3-q}{2q-2}\right)}\sqrt{\frac{q-1}{5-3q}}\left[1 - \left(\frac{1-q}{5-3q}\right)x^2\right]^{\frac{1}{1-q}}. \tag{31}$$

One then sees that the q-Gaussian with zero mean and unity variance has only one free parameter, namely the parameter q.

Assuming that the returns of the underlying asset follows the process (20), Borland [9,10] obtained the following expression for the price C of a European call option:

$$C(S, K, \Delta t; \sigma, q) = SM_q(d_1, d_2, b(\Omega_N)) - e^{-r\Delta t} K N_q(d_1, d_2) \tag{32}$$

where

$$N_q(d_1, d_2) = \frac{1}{Z_N}\int_{d_1}^{d_2} \left(1 - (1-q)\beta_N \Omega_N^2\right)^{\frac{1}{1-q}} d\Omega_N \tag{33}$$

and

$$M_q(d_1, d_2, b(\Omega_N)) = \frac{1}{Z_N}\int_{d_1}^{d_2} e^{b(\Omega_N)}\left(1-(1-q)\beta_N\Omega_N^2\right)^{\frac{1}{1-q}} d\Omega_N, \tag{34}$$

with

$$b(\Omega_N) = \sigma\sqrt{\frac{\beta_N}{\beta(\Delta t)}}\Omega_N - \frac{\sigma^2}{2}\alpha\Delta t^{\frac{2}{3-q}}\left(1-(1-q)\beta_N\Omega_N^2\right). \tag{35}$$

Here β_N is defined as

$$\beta_N = \frac{1}{5-3q}, \tag{36}$$

so that the noise distribution is normalized to unity variance for each value of q [10], whereas Z_N is obtained from (24) as

$$Z_N = \sqrt{\frac{c_q}{\beta_N}}. \tag{37}$$

The limits of integration in the integrals in (33) and (34) are

$$d_{1,2} = \frac{\Omega_{1,2}(\Delta t)}{\sigma\sqrt{\beta_N/\beta(\Delta t)}}, \tag{38}$$

where Ω_1 and Ω_2 are the roots of the following quadratic equation:

$$\sigma\Omega + r\Delta t - \frac{\sigma^2}{2}\alpha\Delta t^{\frac{2}{3-q}}\left(1-(1-q)\beta(\Delta t)\Omega^2\right) - \ln\left(\frac{K}{S}\right) = 0, \tag{39}$$

with

$$\alpha = \frac{1}{2}(3-q)\left[(2-q)(3-q)c_q\right]^{\frac{q-1}{3-q}}. \tag{40}$$

On taking the limit $q \to 1$ in (32), one can verify that the standard Black–Scholes formula is recovered [10].

As seen above, the option price in the q-Gaussian model has two unknown parameters, namely the volatility σ and the parameter q, , whose values are to be determined from a best fit of the option data, as discussed in Section 4. But before, let us first present an empirical analysis of the Ibovespa returns vis-à-vis the three model distributions described above.

3. Empirical analysis of returns

In this section we make a comparative study of the three model distributions described above as applied to the Ibovespa index. We have analyzed two historical series of the Ibovespa: (i) a series of daily closing prices from January 1968 up to February 2004, totaling 8889 data points, and (ii) a series of intraday quotes at every 15 min covering the years from 1998 to 2001, containing 19 995 data points. For both series, we compared the empirical distributions (normalized to unit variance) with the best fit by the three theoretical distributions discussed in Section 2 and thus determined which model is more appropriate in each case.

Fig. 1 shows the empirical distribution of daily returns (circles) and the respective fits by the exponential distribution (thick blue line) and the q-Gaussian (thin red line); also shown for comparison is a Gaussian of unity variance (dashed black line). It is clear from this figure that the empirical distribution deviates quite significantly from a Gaussian. The exponential and q-Gaussian distributions are both in good agreement with the data, but the exponential distribution gives a better fit to the data in the sense that it yields a greater coefficient of determination: $R^2 = 0.9917$ for the exponential distribution, whereas $R^2 = 0.9889$ for the q-Gaussian. We recall that the coefficient of determination R^2 (see, e.g., [24] for a definition) normally ranges from 0 to 1 and indicates how well the data is fitted by the model curve. The closer R^2 is to unity the better is the agreement between the theoretical curve and the empirical data. We note however that strictly speaking the coefficient of determination R^2 is not guaranteed to be in the range from 0 to 1 for nonlinear regression; nonetheless, it is still a relevant figure of merit for the goodness of the fit [24].

In Fig. 1 the exponential distribution was fitted with formula (12) after setting $\delta = 0$ (zero mean) and $\nu = 2/\gamma$ (unity variance), which leaves γ as the only fitting parameter, whereas in fitting the q-Gaussian we used (31) which has q as the only parameter. In both fitting procedures we have used the method of the golden section search [30] to determine the respective values of γ and q that minimize the sum of squared residuals for each distribution. In the former case the fitting parameter was $\gamma = 1.47 \pm 0.01$, which implies from (16) that $\nu = 1.36 \pm 0.01$, thus showing that the adjusted exponential distribution is slightly asymmetric in this case. The fit with the q-Gaussian model yields $q = 1.473 \pm 0.008$ for the daily data.

Fig. 2 shows the empirical distribution of the intraday returns with the same plot convention as in Fig. 1. Here the empirical distribution shows much heavier tails than in the former case. From the visual inspection of Fig. 2 one clearly sees that the q-Gaussian gives a better description of the data in comparison to the exponential distribution. This is confirmed by the fact

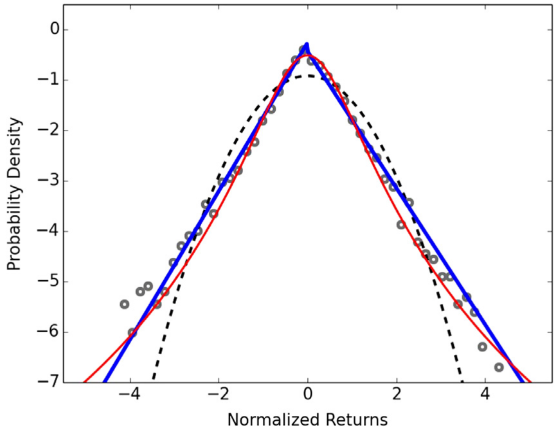

Fig. 1. Empirical distribution of daily returns of the Ibovespa index (circles) and respective fits with exponential (thick blue line) and q-Gaussian (thin red line) distributions. The dashed line represents a Gaussian with unity variance. The fitting parameters are $\gamma = 1.47 \pm 0.01$ (exponential distribution) and $q = 1.473 \pm 0.008$ (q-Gaussian).

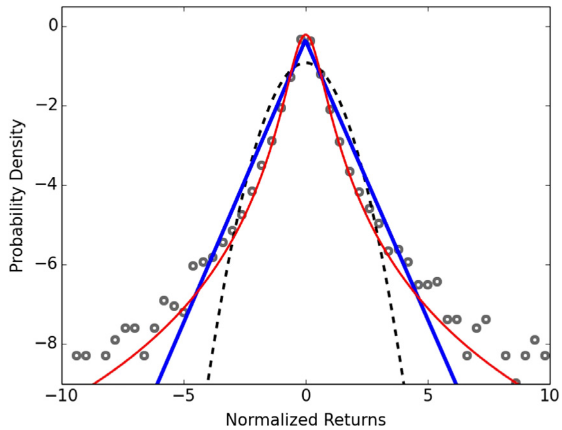

Fig. 2. Empirical distribution of intraday returns of the Ibovespa index at a time lag of 15 min, with the same plot convention as in Fig. 1. Here the fitting parameters are $\gamma = 1.43 \pm 0.04$ and $q = 1.572 \pm 0.002$.

that $R^2 = 0.9982$ for the q-Gaussian, while $R^2 = 0.9359$ for the exponential fit. Here one finds $q = 1.572 \pm 0.002$, which is considerably larger than the value for q obtained for the daily returns, thus showing that the distribution of returns at shorter scales does indeed have more pronounced tails. Evidence of power law tails in the intraday quotes of the Ibovespa has also been observed, e.g., in Refs. [23,31].

As the time lag increases, the power law distribution should converge to a Gaussian, i.e., $q \to 1$, with an exponential distribution appearing at some intermediate crossover scale. In order to study this effect, we have computed the empirical returns at time lags $\tau = 2^n \times 15$ min, for $n = 1, \ldots, 10$, and fitted each data set with a q-Gaussian distribution. Examples of some empirical distributions and respective fits are shown in Fig. 3a for $\tau = 15, 60, 240, 480, 960, 3840$ min. A plot of q as a function of τ is shown in Fig. 3b. One sees from this figure that q decreases toward unity as τ increases, as expected. (Note that the resulting series of returns become increasingly less representative as τ increases, which makes the fitting procedure less reliable for large τ. But nonetheless the trend $q \to 1$ is clearly verified.)

For comparison, we have also fitted the distribution of returns at different time lags τ with the exponential model. In Fig. 4 we plot the respective figure of merit R^2 as a function of lag τ for both the q-Gaussian model (circles) and the exponential model (crosses). One sees from this figure that the exponential model gives a very good fit to the data for scales of the order of a few hours, where its R^2 value is approximately the same as that obtained from the q-Gaussian model (and both very close to 1). As an example, we show in Fig. 3a the fit by the exponential model for $\tau = 240$ min (dashed line). Here the q-Gaussian model performs slightly better than the exponential model for τ of the order of a trading day (7 h), while the opposite occurs for the daily data shown in Fig. 1. This is probably because the intraday series covers a relatively small period of time (three years), which makes the return distributions for larger time lags statistically less significant. Nevertheless, it is evident from the combined analysis of the daily and intraday quotes that the empirical distribution of the Ibovespa at scales from a few hours to a few days is well described by an exponential distribution.

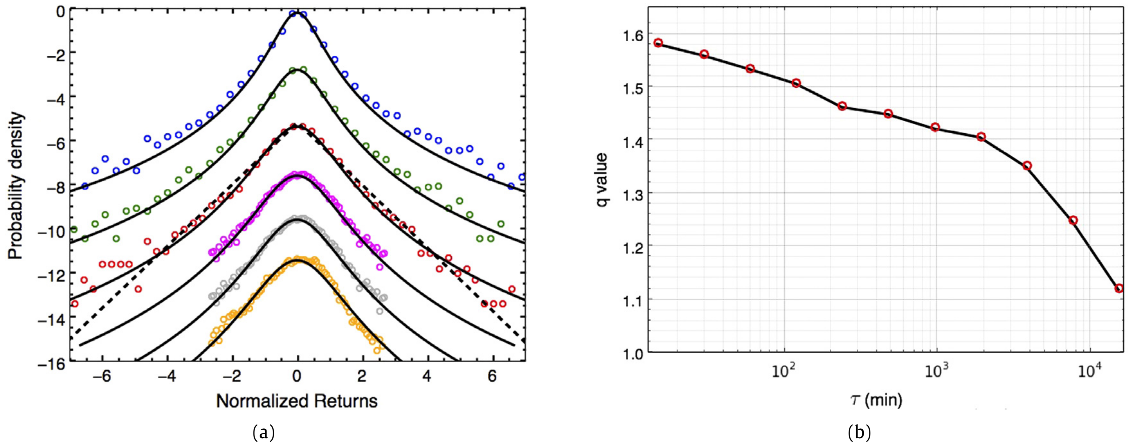

Fig. 3. (a) Distributions of returns (open circles) for time-lags of length $\tau = 15, 60, 240, 480, 960, 3840$ min (from top to bottom) calculated from the intraday quotes at 15 min of the Ibovespa index, together with the respective fits (solid curves) by the q-Gaussian model. The dashed line indicates the fit by the exponential model for $\tau = 240$ min. The curves have been arbitrarily shifted vertically for clarity. (b) Evolution of the q-parameter as a function of the time lag $\tau = 2^n \times 15$ min, for $n = 0, 1, \ldots, 10$.

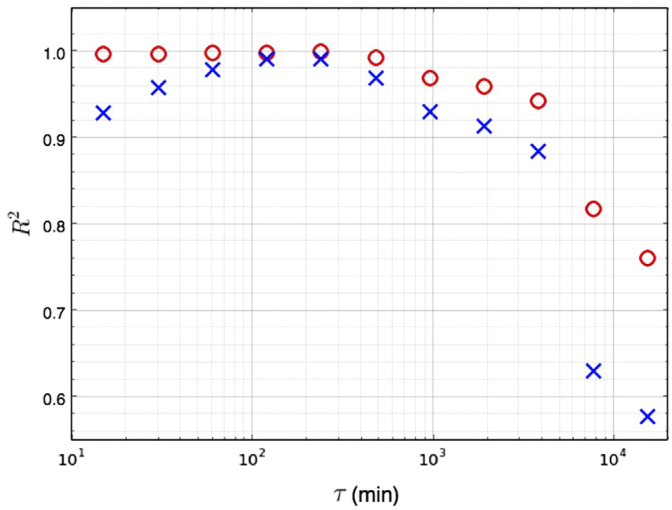

Fig. 4. Figure of merit R^2 for the fits by the q-Gaussian model (circles) and the exponential model (crosses) for the Ibovespa returns at different time lags τ.

The analysis above confirms the fact that the shape of the distribution of the Ibovespa returns varies with the time scale: it exhibits power law tails at short time scales (of the order of minutes or less), follows an exponential decay at mesoscales (from hourly to daily scale), and then tends to a Gaussian at large (monthly) scales. In the next section we investigate the question of pricing options on the Ibovespa in light of this variability.

4. Methodology for options analysis

We have analyzed the daily closing prices of call options on the Ibovespa index during a period of two years (2005–2006). Options that have the same date of expiration constitute what is called an option series. The set of options belonging to the same series that are traded on a particular day is called an *option chain*. In other words, an option chain is a set of option prices (premiums) as a function of the strikes for which trade occurred on a given day.

In the São Paulo Stock Exchange an option series is typically launched 60 days before the expiration date, with trading being authorized on several options with different exercise prices close to the spot price of the underlying asset. However, in the event of future changes in the quoted price of the underlying asset, the issue of options with new strikes for a given series may be authorized, so as to reflect the movement of the asset price. When an option is authorized, it usually remains valid until its maturity. The Exchange may however suspend options that have no open interest and whose exercise prices

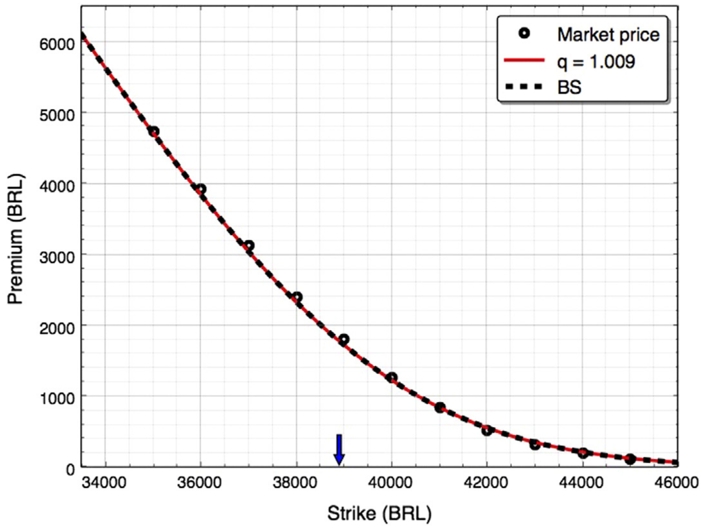

Fig. 5. Market prices (circles) for the option chain of the IBOVL series for year 2006 at 38 days before expiration. Also shown are the respective fits by the Black–Scholes formula (dashed black line) and by the option pricing formula based on the q-Gaussian distribution (solid red line). The arrow indicates the corresponding spot price.

are very different from the asset market price. As a result, the number of strikes traded on a given day for a given option series varies considerably during the lifespan of the series. In particular, near maturity trading tends to concentrate on a small subset of options whose strikes are close to the spot price [24].

Options on the Ibovespa index are of the European type and are denoted by the symbol IBOV followed by a letter and a number. The letter indicates the date of expiration according to the following convention: letters from A to L indicate call options expiring on the months from January to December, respectively. Options on the Ibovespa always expiry on the Wednesday closest to the 15th day of the corresponding month of expiration. The number following the letter corresponds to the exercise price. For instance, the option IBOVL37 of year 2006 denotes the option whose expiration date was December 13, 2006 (the closest Wednesday to December 15, 2006), and whose strike price was BRL 37 000.

In the two-year period analyzed in the present study there were 850 option chains, each containing a number of points (i.e., traded strikes) ranging from 1 to 11. Here, however, we consider only option chains with at least five strikes, so as to make the statistical analysis more significant. This subset contains 345 option chains. Examples of four options chains in this class are shown in Tables 1–4 in the Appendix; these are the option data used in Figs. 5, 6, 10 and 11, corresponding to the closing prices of the respective options on the particular trading days indicated in the Appendix.

To study our set of admissible options (i.e., with more than four strikes) we adopted a methodology similar to that used in Ref. [24], which we briefly summarize here. For a given admissible option chain, we fit the corresponding set of prices C_i versus strike K_i with the theoretical formulas predicted by the models described in Section 2. For instance, in fitting the Black–Scholes formula (8) to a given option chain we assume that the volatility σ is the same for all options in the chain and determine σ by a least-square fit using the golden section search method [30]. Similarly, for the exponential and q-Gaussian models we fit formulas (17) and (32), respectively, to the option data using the Nelder–Mead optimization method [32] to obtain the optimal values of the parameters (γ, ν) and q. In all fits performed here, the risk-free interest rate r was assumed to be the Brazilian Interbank Deposit Rate [33] valid at the time of the option chain.

In comparing the performances of the different models for a given option chain, we look at the respective coefficients of determination R^2 produced by the corresponding best fits. We shall therefore say that a model that yields a higher R^2, i.e., closer to 1, performs better than a model whose best fit has a lower R^2.

5. Empirical analysis of option prices

Here we present a comparative analysis of how the three option pricing models discussed in Section 2 apply to the Brazilian option market. First we compare the performance of the q-Gaussian option model with respect to the standard Black–Scholes model.

We recall that the q-Gaussian distribution recovers the Gaussian for $q = 1$. Thus, if the fit of the option pricing formula (32) for a given option chain returns a value of q that is sufficiently close to unity, i.e., $q \approx 1$, we conclude that the Black–Scholes model describes well this option chain. As a practical rule we adopt the following criterion: if the q-value obtained from the fitting procedure is within the range $1 \leq q < 1.05$, we shall assume that the Black–Scholes model is more suitable in such cases (and effectively take $q = 1$). One example of this case is given in Fig. 5, where we show the market prices

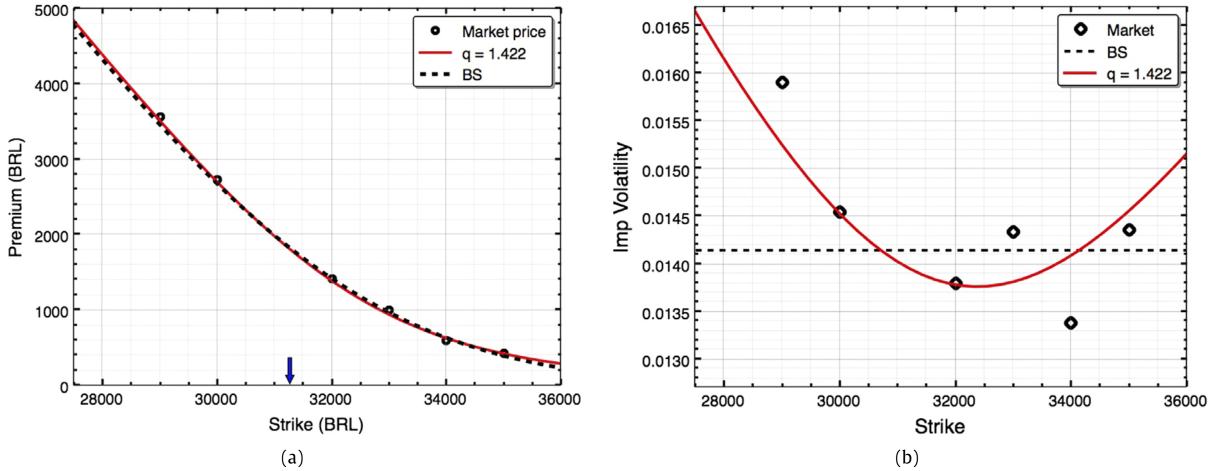

Fig. 6. (a) Market prices (circles) for the option chain of the IBOVL series for year 2005 at 49 days before the expiration date, together with fits by the Black–Scholes and q-Gaussian models. (b) Corresponding implied volatilities for the option chain shown in panel (a).

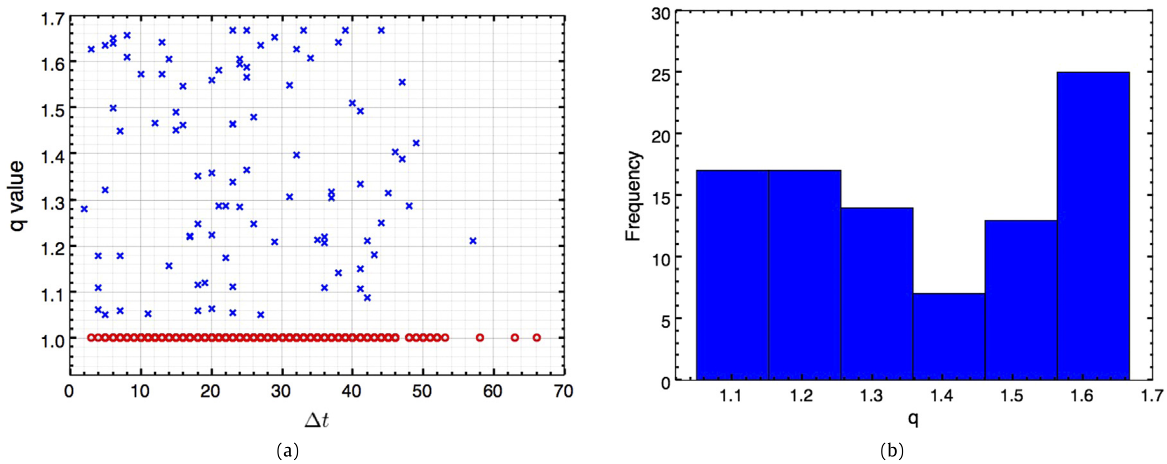

Fig. 7. (a) Values of the q parameter as a function of the time Δt to expiration. Values with $q > 1$ are indicated by blue crosses and $q = 1$ by red circles. (b) Corresponding histogram of q values for the cases where $q > 1$.

(circles) for the option chain belonging to the IBOVL series for the year 2006 at 38 days prior to maturity. Here the fit by the q-Gaussian model (red solid line) yields $q = 1.009$, meaning that in practice the Black–Scholes formula works better than the general formula for any $q > 1$. Also shown for comparison in Fig. 5 is the fit obtained by directly employing the Black–Scholes formula (dashed line), which is indeed indistinguishable from the fit obtained using the general formula based of the q-Gaussian model.

If a value $q > 1$ is obtained (according to the criterion above) for a given option chain, we then conclude that in this case it is more advantageous to use the q-Gaussian model as it fits better the data than the Black–Scholes formula ($q = 1$). One example of this case is presented in Fig. 6a where we show the option chain belonging to the IBOVL series for year 2005 at 49 days prior to maturity. The best fit by the q-Gaussian model (red solid line) yields $q = 1.422$. For comparison, we also show in Fig. 6a the fit with the Black–Scholes formula (dashed line). Although both models agree reasonably well with the empirical data, the q-Gaussian model provides a slightly better fit, which is confirmed by comparing the respective coefficients of determination: $R^2 = 0.9992$ for the q-Gaussian model, whereas and $R^2 = 0.9984$ for the Black–Scholes model. Even though both models give a value of R^2 very close to one another (and close to unity), it is nonetheless legitimate to discern between the two models on the basis of this figure of merit; see, e.g., Ref. [24] for further discussion on this point.

An alternative way to display the results presented in Fig. 6a is shown in Fig. 6b, where we plot the corresponding implied volatilities for the empirical data (open circles), the Black–Scholes model (black dashed line) and the q-Gaussian model (red solid curve). It is clear from this figure that the q-Gaussian model adjusts better the implied volatilities, and so it does indeed give a better description of the market option prices in comparison with the Black–Scholes model.

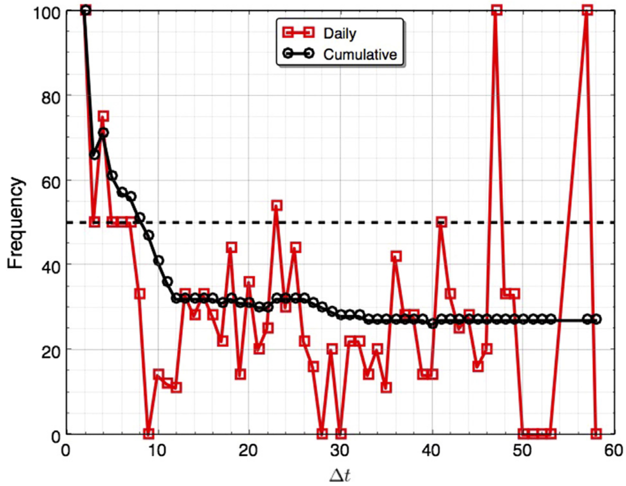

Fig. 8. Percentage of option chains fitted by the q-Gaussian model with $q > 1$ as a function of the time Δt to maturity. The red squares are the results for each Δt and the black circles are the cumulative frequency from 0 to Δt; see text.

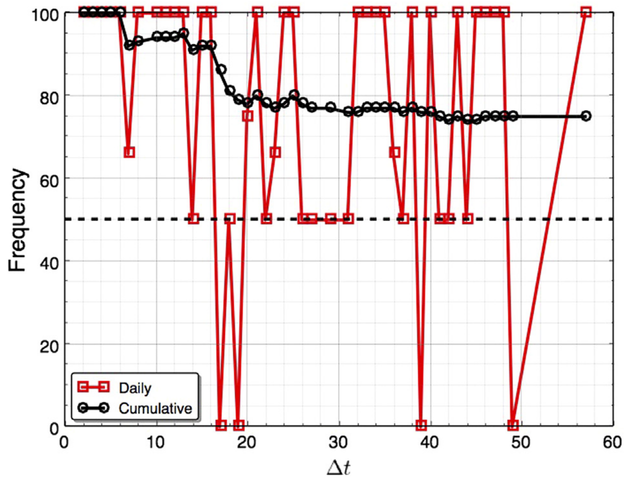

Fig. 9. Percentage of option chains better fitted by the exponential model in comparison to the q-Gaussian model as a function of the time Δt to maturity. The red squares are the results for each Δt and the black circles are the cumulative frequency from 0 to Δt.

In order to make a more extensive comparison between the Black–Scholes and the q-Gaussian models, we have performed least-square fits of formula (32) for all 345 admissible option chains in our data set. In Fig. 7a we show the values of q as a function of the time Δt to expiration (in days). Values with $q > 1$ are denoted by blue crosses, whereas the cases where $q = 1$ (according to the criterion above) are indicated in red circles. In Fig. 7b we show the histogram corresponding to the values $q > 1$ obtained in Fig. 7a. Although larger values of q (implying heavier tails) tend to be favored – notice that the mode lies in the interval $[1.56, 5/3)$ –, there is nonetheless a considerable spread in the values of $q > 1$. This implies that, in general, different option chains may require different values of q. This in contrast with the approach adopted in [9,10] where the value of q is estimated from the return distribution of the underlying asset and hence it is assumed to be the same for all options on this asset.

To make a comparative analysis of the performance of the two models as a function of the time to maturity, we plot in Fig. 8 the percentage of cases with $q > 1$ among all option chains for each Δt (red squares). The dashed horizontal line in the figure corresponds to the 50% line. One sees from this plot that the q-Gaussian model tends to perform better than the Black–Scholes model for options close to maturity, since for $\Delta t < 9$ days the points (squares) lie above or at the 50% line. This trend is more clearly seen when one looks at the cumulative frequency of the cases with $q > 1$, as shown by the black circles in Fig. 8. This curve gives the percentage of the cases with $q > 1$ among all option chains with a time to maturity less than or equal to a given Δt. One sees from this plot that the q-Gaussian model performs better in the majority of the cases for all times up to 8 days prior to maturity, as the cumulative frequency remains above 50% up to this point. For larger

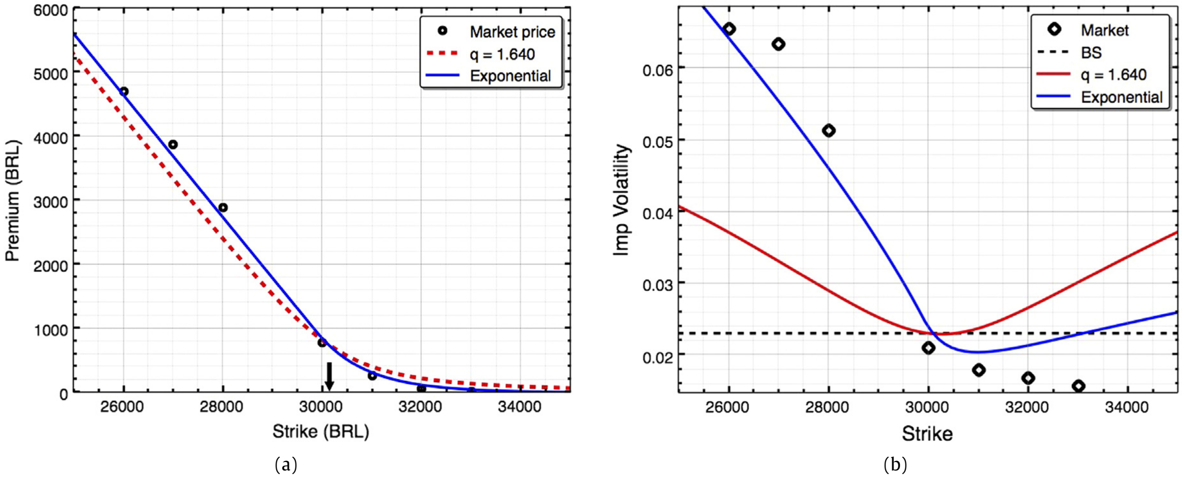

Fig. 10. (a) Comparison between best fits to the market option prices (circles) by the q-Gaussian model (red dashed line) and the exponential model (solid blue line) for the option chain of the IBOVJ series for year 2005 at 6 days before expiration. The arrow indicates the corresponding spot price. (b) Corresponding implied volatilities for the empirical data (circles), the q-Gaussian model (red line), and the exponential model (blue line).

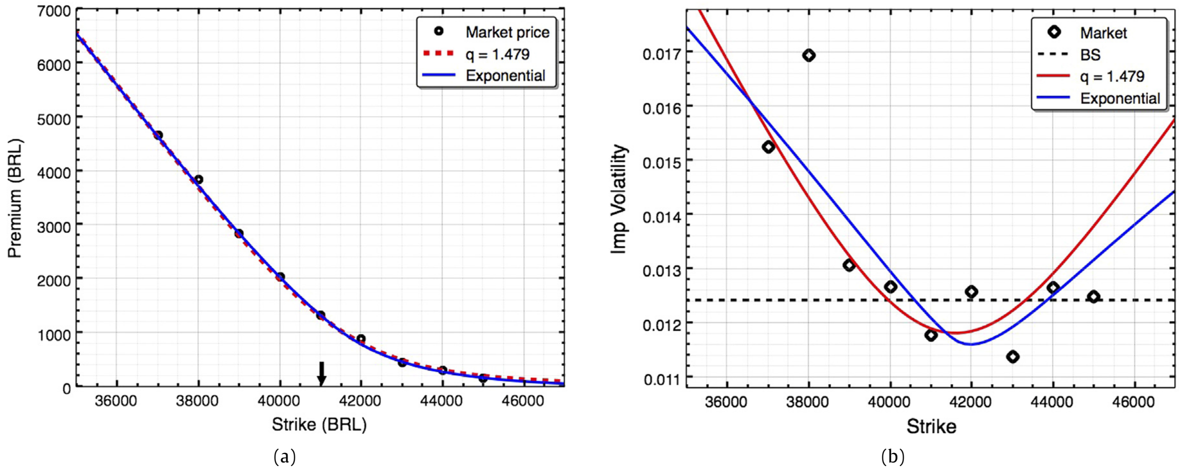

Fig. 11. (a) Best fits to the market option prices (circles) by the q-Gaussian model (red dashed line) and the exponential model (solid blue line) for the option chain of the IBOVL series of year 2006 at 26 days before expiration. (b) Corresponding implied volatilities for the empirical data (circles), the q-Gaussian model (red line), and the exponential model (blue line).

Δt the Black–Scholes model adjusts better the data in the majority of cases. This analysis shows furthermore that among all options chains analyzed the q-Gaussian model (with $q > 1$) gives a better fit to the data, as compared to the Black–Scholes formula, in only 27% of the cases; see the last circle in Fig. 8.

For the cases where the q-Gaussian model yields $q > 1$, thus surpassing the Black–Scholes model, we have investigated how it compares to the exponential model. To this end, we have also fitted the option chains in this class ($q > 1$) with the price formula from the exponential model and compared the corresponding fits by both models. The result of this comparison is shown in Fig. 9, where we plot the percentage of cases for which the exponential model gives a better fit to the option data. As in Fig. 8, the red squares correspond to the frequency for each Δt, whereas the black circles are the cumulative frequency from 0 to Δt. One first result from this analysis is that the exponential model adjusts better the option data in 75% of all cases; see last circle in Fig. 9. The superiority of the exponential model over the q-Gaussian model is particularly striking for option chains near maturity. For instance, one sees from Fig. 9 that the exponential model fits better the data in 90% of all options chains with less than 16 days to expiration and in 100% of the cases within 6 days to maturity or less.

One example of an option chain very close to maturity is shown in Fig. 10a for the IBOVJ series of year 2005 at 6 days before expiration. Here one sees that the exponential model does indeed provide a much better fit to the option prices than the q-Gaussian model. Notice, in particular, that the q-Gaussian model performs rather poorly for in-the-money options, whereas the exponential model seems to have more 'flexibility' in that it describes well both in-the-money and out-of-the-money options. This flexibility of the exponential model is also clearly seen in Fig. 10b, where we plot the market implied volatilities together with the respective implied volatility computed from both models. Notice that the market implied volatility is highly asymmetrical in this case. In contrast to the q-Gaussian model which is symmetric, the exponential model is non-symmetric by definition, see Eq. (12), and hence it is more capable of describing the skewed volatility smile.

For option chains farther from expiration the discrepancy between the two models is in general less pronounced, as illustrated in Fig. 11a for the IBOVL series of year 2006 at 26 days prior to maturity. Here both models provide reasonably good fits to the empirical data, with the exponential model performing slightly better in the sense that it yields a higher coefficient of determination, namely $R^2 = 0.9991$, as compared to $R^2 = 0.9989$ for the q-Gaussian model. In this case the volatility smile has a more symmetrical pattern, as shown in Fig. 11b, thus explaining why both models give reasonably good descriptions of the market option prices. Overall, however, the exponential model has a much better performance than the q-Gaussian model, as discussed above.

6. Discussion and conclusion

We have performed an empirical analysis of the Brazilian market option in light of three option pricing models, namely the standard Black–Scholes model, which assumes that the returns are Gaussian distributed, and two non-Gaussian models based on the exponential and the q-Gaussian distributions, respectively. As the q-Gaussian distribution (with $q > 1$) has power law tails – a common feature of financial data –, particular emphasis was given to the comparison between this model and the other two models whose underlying distributions have exponential decays.

First we analyzed how the q-Gaussian model compares to the Black–Scholes model ($q = 1$). This comparison was made by fitting the pricing formula given by the q-Gaussian model to each option chain in our database and then analyzing the returned q-value for each case. If the best fit yields $q \approx 1$ for a given option chain, it means that the Black–Scholes describes well the corresponding empirical data, whilst a value of $q > 1$ indicates that the q-Gaussian model better fits the data. We found that the value $q = 1$ is observed in over 70% of all option chains analyzed, thus meaning that in only about 30% of the cases does the q-Gaussian model (with $q > 1$) represent an improvement with respect to the Black–Scholes formula.

For the cases when $q > 1$, we then compared the q-Gaussian model with the exponential model. Here we found that in only 25% of these cases does the q-Gaussian model give a better fit to the data than the exponential model. The two results above combined thus show that the overall performance of the q-Gaussian model is rather poor, for in only about 7% of all option chains analyzed it perform simultaneously better than both the Black–Scholes and the exponential models.

We have found, in particular, that for options near maturity the exponential model performs much better than the q-Gaussian model. For example, we found that the former model fits better the empirical data for *all* option chains within 6 days to maturity. A previous comparative study of the exponential and Black–Scholes models [24] revealed a similar trend, with the exponential model performing better than the Black–Scholes model for times closer to the expiration date. These two results, in combination, thus show that for options near maturity the exponential model gives the best overall description of the option prices. This finding in is line with the observation that the empirical distribution of the daily Ibovespa returns is better described (particularly in its central part) by an exponential distribution.

In light of the results above, one is led to conclude that option pricing models based on power-law distributions appear to be less relevant for real markets. This might be related to the fact that heavy-tailed distributions usually occur at shorter time scales as compared to the typical option trading frequency. Another reason for the poor performance of the q-Gaussian model in comparison with the exponential model is related to the fact that the market implied volatility for option chains close to maturity exhibits a strongly nonsymmetric pattern as a function of the strike price—i.e, the volatility smile is skewed. The q-Gaussian model, being symmetric, cannot describe such asymmetric patterns, whereas the exponential model is asymmetric by definition and hence can cope with skewed volatility smiles.

Although we have restricted our study of the Brazilian option market to the years 2005 and 2006, we believe that our main result, namely that options near maturity are better described by the exponential model, should hold in present times. In this regard, it is perhaps worth pointing out that the Brazilian stock market has become relatively more efficient after the liberalization reforms of the early 1990's [34], although short periods of increased inefficiency may eventually occur [35]. The statistical features of the Ibovespa index should thus be somewhat robust in recent times, and so the results reported here are expected to be of general validity.

Here we have analyzed only options on a stock index, namely the Ibovespa index. It would be interesting to perform a similar analysis of options on individual stocks to verify whether the exponential model applies to these options as well. For example, options on the stocks of Petrobras and Vale (two of the largest companies listed on the São Paulo Stock Exchange) are very liquid and hence are natural candidate for future investigation. The results described here should also apply to other financial markets where exponential distributions have been observed [20–22,25].

Another important point for further studies concerns the question of developing investment strategies based on the exponential model which may allow one to trade options more efficiently. Such trading strategies should be particularly relevant near the expiration date, since in this case the exponential model has been shown to perform significantly better than the two other models analyzed here.

As already mentioned, we have found evidence that the superiority of the exponential model over the q-Gaussian model seems to stem from the fact that the market implied volatility smile is skewed to the left—a feature that cannot be captured

Table 1
Option chain of the IBOVL series traded on October 19, 2006, corresponding to 38 days before expiration. Here we show the option price C as a function of the strike price K. The Ibovespa spot price on this day was $S = 38\,919.75$. All prices are in Brazilian reals (BRL).

K	35 000	36 000	37 000	38 000	39 000	40 000	41 000	42 000	43 000	44 000	45 000
C	4720	3904	3109	2387	1800	1250	835	502.5	305	180	100

Table 2
Option chain of the IBOVL series on traded on October 04, 2005, 49 days before expiration, at which date $S = 31\,283.83$.

K	29 000	30 000	32 000	33 000	34 000	35 000
C	3560	2715	1400	995	580	415

Table 3
Option chain of the IBOVJ series traded on October 05, 2005, 6 days before expiration. Here $S = 30\,163.52$.

K	26 000	27 000	28 000	30 000	31 000	32 000	33 000
C	4680	3866	2870	770	250	52.5	5

Table 4
Option chain of the IBOVL series traded on November 07, 2006, 26 days before expiration. Here $S = 41\,048.30$.

K	37 000	38 000	39 000	40 000	41 000	42 000	43 000	44 000	45 000
C	4647.5	3830	2820	2030	1315	880	435.4	293.5	145

by the symmetric q-Gaussian model. It would be interesting therefore to compare the exponential model with other option pricing models with skew, such as the constant elasticity of variance model [36,37] and the q-Gaussian model with skew [38]. It has been observed, however, that the q-Gaussian model allows arbitrage, and a corrected version has been proposed [39] but where explicit pricing formulas are no longer available. It would nevertheless be interesting to include this modified model as well in future studies. It would also be useful to consider other non-Gaussian option pricing models, such as models based on the stretched exponential distribution [5,6] and nonparametric pricing strategies based on the minimization of risk [16,17]. Such an extended comparison between different option models will be left for future publications.

Acknowledgments

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Appendix. Option chains data

See Tables 1–4.

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